Extra Practice Problems 1

This handout contains a bunch of practice problems you can use to improve your skills and generally get prepped for the upcoming midterm exam. As always, you are encouraged to ask questions if you have them – we're happy to help out!

Problem One: First-Order Logic

Here's some more practice with translating statements into first-order logic. Each of these logic translations appeared on some past midterm exam.

i. Given the predicates

   Person(p), which states that p is a person, and
   ParentOf(p_1, p_2), which states that p_1 is the parent of p_2,

   write a statement in first-order logic that says "someone is their own grandparent." (Paraphrased from an old novelty song.)

ii. Given the predicates

   Set(S), which states that S is a set, and
   $x \in y$, which states that x is an element of y,

   write a statement in first-order logic that says "for any sets S and T, the set $S \Delta T$ exists."

iii. Given the predicates

   Set(S), which states that S is a set;
   $x \in y$, which states that x is an element of y;
   Natural(n), which states that n is a natural number; and
   $x < y$, which states that x is less than y,

   write a statement in first-order logic that says "if S is a nonempty subset of $\mathbb{N}$, then S contains a natural number that's smaller than all the other natural numbers in S." (This is called the ***well-ordering principle***.)

Problem Two: First-Order Negations

For each of the statements you came up with in part (i) of this problem, negate that statement and push the negation as deep as possible, along the lines of what you did in Problem Set Two. Then, for each statement, translate it back into English and make sure you see why it's the negation of the original formula.

Problem Three: Properties of Sets

Below are a number of claims about sets. For each claim, decide whether the statement is true or false. If it's true, prove it. If it's false, disprove it.

i. For all sets A and B, the following is true: $(A - B) \cup B = A$.
ii. For all sets A, B, and C, if $A \subseteq B \cap C$, then $A \subseteq B$ and $A \subseteq C$.
iii. For any set A, if $A \subsetneq \emptyset$, then $137 \in A$.

Problem Four: Symmetric Latin Squares

A ***Latin square*** is an $n \times n$ grid filled with the numbers 1, 2, 3, …, n such that every number appears in every row and every column exactly once. For example, the following are Latin squares:

1	2	3
3	1	2
2	3	1

4	2	1	3
1	3	2	4
3	1	4	2
2	4	3	1

1	3	5	2	4
2	4	1	3	5
3	5	2	4	1
4	1	3	5	2
5	2	4	1	3

A ***symmetric Latin square*** is a Latin square that is symmetric across the main diagonal (the one from the upper-left corner to the lower-right corner). That is, the elements at positions (i, j) and (j, i) are always the same. For example:

1	2	3
2	3	1
3	1	2

4	2	3	1
2	3	1	4
3	1	4	2
1	4	2	3

1	2	3	4	5
2	4	5	3	1
3	5	2	1	4
4	3	1	5	2
5	1	4	2	3

Prove that in any $n \times n$ symmetric Latin square where n is odd, every number 1, 2, 3, …, n must appear at least once on the main diagonal. As a hint, split the Latin Square into three regions – the main diagonal, the triangle above the main diagonal, and the triangle below the main diagonal.

Problem Five: The Logic of Elections

Two candidates X and Y are running for office and are counting final votes. Candidate X argues that more people voted for them than for Candidate Y by making the following claim: "For every ballot cast for Candidate Y, there were two ballots cast for Candidate X." Candidate X states this in logic as follows:

$$
\begin{aligned}
&\forall b.\ (BallotForY(b) \rightarrow \\
&\quad \exists b_1.\ \exists b_2.\ (BallotForX(b_1) \land BallotForX(b_2) \land b_1 \neq b_2) \\
&)
\end{aligned}
$$

However, it is possible for the above first-order logic statement to be true even if Candidate X didn't get the majority of the votes. Give an example of a set of ballots such that

1. every ballot is cast for exactly one of Candidate X and Candidate Y,
2. the set of ballots obeys the rules described by the above statement in first-order logic, but
3. candidate Y gets strictly more votes than Candidate X.

You should justify why your set of ballots works, though you don't need to formally prove it. Make specific reference to the first-order logic statement in your justification.

Problem Six: The Minkowski Sum

If $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$, then the ***Minkowski sum*** of A and B, denoted $A + B$, is the set

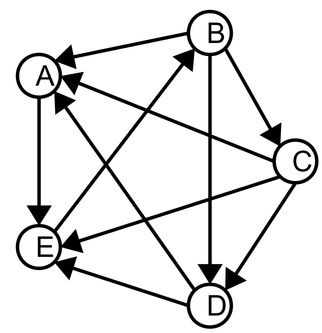
$$A + B = \{ m + n \mid m \in A \text{ and } n \in B \}$$

This question explores properties of the Minkowski sum.

i. Prove or disprove: $|A + B| = |A| \cdot |B|$ for all finite sets $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$.
ii. What is $\mathbb{N} + \mathbb{N}$? Prove it.

Problem Seven: Tournaments

Recall from Problem Set Two that ***tournament*** is a contest among n players. Each player plays a game against each other player, and either wins or loses the game (let's assume that there are no draws). We can visually represent a tournament by drawing a circle for each player and drawing arrows between pairs of players to indicate who won each game. For example, in the tournament to the left, player A beat player E, but lost to players B, C, and D.



A ***tournament champion*** is a player in a tournament who, for each other player, either won her game against that player, or won a game against a player who in turn won his game against that player (or both). In the tournament to the left, players B, C, and E are tournament champions. However, player D is not a tournament champion, because he neither won against player C, nor won against anyone who in turn won against player C. Although player D won against player E, who in turn won against player B, who then won against player C, under our definition player D is not a tournament champion.

i. Let n be an arbitrary odd natural number. An ***egalitarian tournament*** with $n = 2k + 1$ players is one where every player won exactly k games. Prove that every player in an egalitarian tournament is a tournament champion
ii. If T is a tournament and p is a player in T, then let $W(p) = \{ q \mid q \text{ is a player in } T \text{ and } p \text{ beat } q \}$. Prove that if p_1 and p_2 are players in T and $p_1 \neq p_2$, then $W(p_1) \neq W(p_2)$.

Let's introduce one more definition. A ***pseudotournament*** is like a tournament, except that exactly one pair of people don't play a game against one another.

iii. Prove that for any $n \geq 2$, there's a pseudotournament with n players and no champions.
iv. Is your result from (ii) still valid if T is just a pseudotournament, rather than a full tournament?

Problem Eight: Hungry Logic

(From the Fall 2015 midterm exam)

Let's imagine that you're *really* hungry and want to build an infinitely tall cheese sandwich. Your sandwich will consist an infinite alternating sequence of slices of bread and slices of cheese.

Using the predicates

Bread(b), which states that b is a piece of bread;
Cheese(c), which states that c is a piece of cheese; and

Atop(x, y), which says that x is directly on top of y,

write a statement in first-order logic that says "every piece of bread has a piece of cheese directly on top of it, every piece of cheese has a piece of bread directly on top of it, and there's at least one piece of bread."

Problem Nine: Propositional Logic

Below are a series of English descriptions of relations among propositional variables. For each description, write a propositional formula that precisely encodes that relation. Then, briefly explain the intuition behind your formula. You may find the online truth table tool useful here.

i. For the variables a, b, c, and d: the variables, written out in alphabetical order, alternate between true and false.

ii. For the variables a, b, c, and d: the variables, written out in alphabetical order, alternate between true and false, except that your formula cannot use the $\vee$ connective.

Problem Ten: More Modular Arithmetic

Here's a few more properties of the modular congruence relation to explore! In this problem, assume all variables represent integers.

i. Prove that if $w \equiv_k y$ and $x \equiv_k z$, then $w + x \equiv_k y + z$.

ii. Prove that if $w \equiv_k y$ and $x \equiv_k z$, then $wx \equiv_k yz$.

Problem Eleven: Slicing an Orange

You have a perfectly spherical orange with five stickers on it. Prove that there is some way to slice the orange into two equal halves so that one of the halves has pieces of at least four of the stickers on it.

Problem Twelve: Inductive Sets

A set S is called an ***inductive set*** if the follow two properties are true about S:

- $0 \in S$.
- For any number $x \in S$, the number $x + 1$ is also an element of S.

This question asks you to explore various properties of inductive sets.

i. Find two different examples of inductive sets.

ii. Prove that the intersection of any two inductive sets is also an inductive set.

iii. Prove that if S is an inductive set, then $\mathbb{N} \subseteq S$.

iv. Prove that $\mathbb{N}$ is the *only* inductive set that's a subset of all inductive sets. This proves that $\mathbb{N}$ is, in a sense, the most "fundamental" inductive set. In fact, in foundational mathematics, the set $\mathbb{N}$ is sometimes defined as "the one inductive set that's a subset of all inductive sets." (Take Math 161 for details!)

Problem Thirteen: Odd and Even Functions

Up to this point, most of our discussion of functions has involved functions from arbitrary domains to arbitrary codomains. If we restrict ourselves to functions with specific types of domains and codomains, then we can start exploring more nuanced and interesting classes of functions.

Let's suppose that we have a function $f : \mathbb{R} \to \mathbb{R}$. We'll say that f is an ***odd function*** if the following is true:

$$\forall x \in \mathbb{R}.\ f(-x) = -f(x)$$

This function explores properties of odd functions.

i. Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are odd, then $g \circ f$ is also odd.

ii. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is odd and is a bijection, then f^{1} is also odd.

We can define ***even functions*** as follows. A function $f : \mathbb{R} \to \mathbb{R}$ is called ***even*** if the following is true:

$$\forall x \in \mathbb{R}.\ f(-x) = f(x)$$

iii. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function, then f is ***not*** a bijection.

It turns out that every function $f : \mathbb{R} \to \mathbb{R}$ can be written as the sum of an odd function and an even function. The next few parts of this problem ask you to prove this.

iv. Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function. Prove that for any $r \in \mathbb{R}$, the function $r \cdot f : \mathbb{R} \to \mathbb{R}$ defined as $(r \cdot f)(x) = r \cdot f(x)$ is also odd.

v. Let $f : \mathbb{R} \to \mathbb{R}$ be an even function. Prove that for any $r \in \mathbb{R}$, the function $r \cdot f : \mathbb{R} \to \mathbb{R}$ defined as $(r \cdot f)(x) = r \cdot f(x)$ is also even.

vi. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Prove that $g : \mathbb{R} \to \mathbb{R}$ defined as $g(x) = f(x) - f(-x)$ is odd.

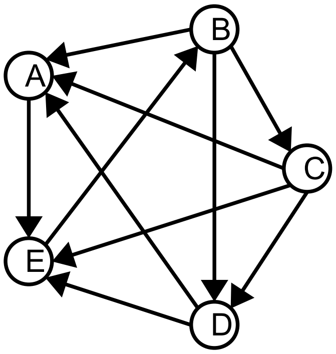
vii. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Prove that $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) = f(x) + f(-x)$ is even.

viii. Prove that any function $f : \mathbb{R} \to \mathbb{R}$ can be expressed as the sum of an odd function and an even function.

Problem Fourteen: Tournament Graphs and Binary Relations

Let's quickly refresh a definition. A ***tournament*** is a contest between some number of players in which each player plays each other player exactly once. We assume that no games end in a tie, so each game ends in a win for one of the players.

Here's a new definition. If p is a player in tournament T, we define $W(p) = \{ x \mid x \text{ is a player in } T \text{ and } p \text{ beat } x \}$. Intuitively, $W(p)$ is the set of all the players that player p beat. For example, in the tournament on the left, $W(B) = \{A, C, D\}$.



Now, let's define a new binary relation. Let T be a tournament. We'll say that $p_1 \sqsubset_T p_2$ if $W(p_1) \subsetneq W(p_2)$. Intuitively, $p_1 \sqsubset_T p_2$ means that p_2 beat every player that p_1 beat, plus some additional players.

For example, in the tournament to the left, we have that $D \sqsubset_T C$ because $W(D) = \{A, E\}$ and $W(C) = \{A, D, E\}$. Similarly, we know $A \sqsubset_T D$ since $W(A) = \{E\}$ and $W(D) = \{A, E\}$.

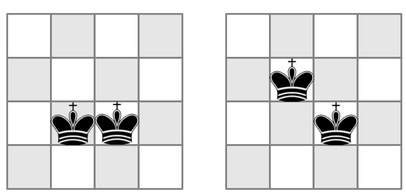
Prove that if T is a tournament, then $\sqsubset_T$ is a strict order over the players in T.

Problem Fifteen: Tournament Graphs and Hamiltonian Paths

A ***tournament graph*** is a directed graph of n nodes where every pair of distinct nodes has exactly one edge between them. A ***Hamiltonian path*** is a path in a graph that passes through every node in a graph exactly once. Prove that every tournament graph has a Hamiltonian path. For the purposes of this problem you can consider the *empty path* of no nodes to be a Hamiltonian path through the empty graph.

Problem Sixteen: A Clash of Kings

Chess is a game played on an 8 × 8 grid with a variety of pieces. In chess, no two king pieces can ever occupy two squares that are immediately adjacent to one another horizontally, vertically, or diagonally. For example, the following positions are illegal:



Prove that it is impossible to legally place 17 kings onto a chessboard.

Problem Seventeen: Induction and Strict Orders

Let A be a set and $<_A$ be a strict order over A. A new definition: a ***chain in*** $<_A$ is a series of elements $x_1, \ldots, x_n$ drawn from A such that

$$x_1 <_A x_2 <_A \ldots <_A x_n.$$

Prove, by induction, that if $x_1, \ldots, x_n$ is a chain in $<_A$ with $n \geq 2$ elements, then $x_1 <_A x_n$.

Problem Eighteen: Strengthening Relations

Let's introduce a new definition. Let R and T be binary relations over the same set A. We'll say that R is ***no stronger than*** than T if the following statement is true:

$$\forall a \in A.\ \forall b \in A.\ (aRb \to aTb)$$

i. Let R and T be binary relations over the same set A where R is no stronger than T. Prove or disprove: if R is an equivalence relation, then T is an equivalence relation.

ii. Let R and T be binary relations over the same set A where R is no stronger than T. Prove or disprove: if T is an equivalence relation, then R is an equivalence relation.

Problem Nineteen: More Fun With Friends and Strangers

(From the Fall 2013 midterm exam)

Suppose you have a 17-clique (that is, an undirected graph with 17 nodes where there's an edge between each pair of nodes) where each edge is colored one of *three* different colors (say, red, green, and blue). Prove that regardless of how the 17-clique is colored, it must contain a blue triangle, a red triangle, or a green triangle. As a hint, use the Theorem on Friends and Strangers.

Problem Twenty: Bijections and Induction

(From the Fall 2014 midterm exam)

Let $f : \mathbb{N} \to \mathbb{N}$ be a function. We'll say that f is ***linearly bounded*** if $f(n) \leq n$ for all $n \in \mathbb{N}$.

Prove that if $f : \mathbb{N} \to \mathbb{N}$ is linearly bounded and is a bijection, then $f(n) = n$ for all $n \in \mathbb{N}$. As a hint, you may want to use induction.

A good question to ponder: is this result still true if we replace the codomain of $\mathbb{N}$ with $\mathbb{Z}$? If so, why? If not, why not? And if the answer is no, what specific claim can you point to in your proof that is no longer true?

Problem Twenty One: Odd Rational Numbers

Let's begin with some new definitions. First, we say that a real number x is a ***rational number*** if there are integers p and q where $q \neq 0$ and $x = {}^{p}/_{q}$. For example, 1.7 = 17 / 10 is a rational number. Next, we'll say that a real number r is an ***odd rational number*** if there exist integers p and q where $r = {}^{p}/_{q}$ and q is odd. For example, the number 1.6 is an odd rational number because it can be written as ${}^{8}/_{5}$.

i. To help you get more familiar with the definition, prove that ${}^{3}/_{2}$ is not an odd rational number. *(Hint:* ***Read the definition closely****. What exactly do you need to prove here?)*

Consider the following binary relation ~ over the set $\mathbb{R}$:

$$x \sim y \quad \text{if} \quad y - x \text{ is an odd rational number.}$$

ii. Prove that ~ is an equivalence relation.

iii. What is $[0]_{\sim}$? Express your answer as simply as possible.

Problem Twenty Two: Long Paths

(From the Fall 2016 midterm exam)

Let $G = (V, E)$ be a graph where every node has degree at least k for some $k \geq 1$. Let P be a simple path in G that has length less than k. Prove that P is ***not*** the longest simple path in G.

Problem Twenty Three: Least and Greatest Elements

Let $<_A$ be a strict order over a set A. We say that an element x is a ***least element of*** $<_A$ if for every element $y \in A$ other than x, the relation $x <_A y$ holds. We say that an element x is a ***greatest element of*** $<_A$ if for every element $y \in A$ other than x, the relation $y <_A x$ holds.

i. Give an example of a strict order relation with no least or greatest element. Briefly justify your answer.

ii. Give an example of a strict order relation with a least element but no greatest element. Briefly justify your answer.

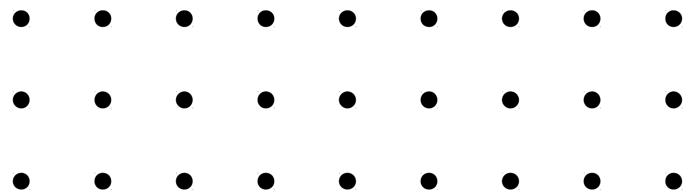
iii. Give an example of a strict order relation with a greatest element but no least element. Briefly justify your answer.

iv. Give an example of a strict order with a greatest element and a least element. Briefly justify your answer.

v. Prove that every strict order has at most one greatest element.

Problem Twenty Four: Coloring a Grid

You are given a 3 × 9 grid of points, like the one shown below:



Suppose that you color each point in the grid either red or blue. Prove that no matter how you color those points, you can always find four points of the same color that form the corners of a rectangle.

A good follow-up question: is a 3 × 9 grid the smallest grid that guarantees a rectangle?

Problem Twenty Five: Trees

Recall from lecture that a ***tree*** is an undirected, connected graph with no cycles.

A ***leaf*** in a tree is a node in a tree whose degree is exactly one.

i. Prove that any tree with at least two nodes has at least one leaf.
ii. In lecture, we used complete induction to prove that any tree with $n \geq 1$ nodes has exactly n-1 edges. Using your result from part (i), prove this result using only standard induction.

Problem Twenty Six: Lattice Points

A ***lattice point*** in 2D space is a point whose (x, y) coordinates are integers. For example, (137, -42) is a lattice point, but (1.5, π) isn't.

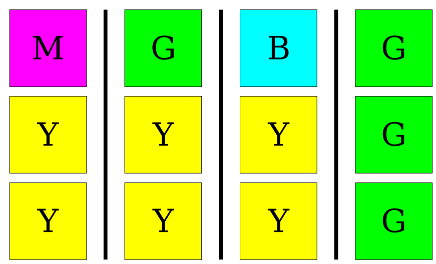
Suppose that you pick any five lattice points in 2D space. Prove that there must be some pair of points in the group with the following property: the midpoint of the line connecting those points is also a lattice point.

Problem Twenty Seven: Colored Cubes*

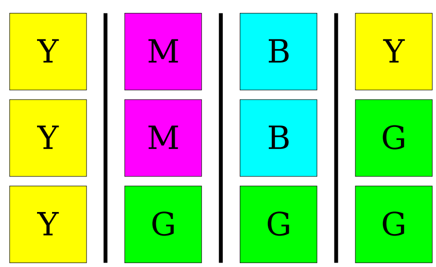
Suppose that you have a collection cubes of n different colors. For simplicity we'll assume that the total number of cubes you have is a multiple of n; specifically, let's suppose that you have kn total cubes, where k is some natural number. For example, you might have 30 cubes of six different colors, in which case $n = 6$ and $k = 5$. Alternatively, you might have 200 cubes of 40 different colors, where $n = 40$ and $k = 5$.

Now, let's suppose that you have n bins into which you can place the cubes, each of which holds exactly k different cubes. Although it may not seem like it, it's always possible to distribute the cubes into the boxes such that every box is full (that is, it has exactly k cubes in it) and that each box has cubes of at most two different colors. Prove this fact using induction on n, the number of colors.

Some examples might help here. Suppose that $n = 4$ and $k = 3$, meaning that there are four different colors of cubes, twelve total cubes, and four boxes that hold three cubes each. The goal is then to put the cubes into the four boxes such that every box has exactly three cubes and contains cubes of at most two different colors. If you have six yellow (Y) cubes, four green (G) cubes, one blue (B) cube, and one magenta (M) cube, here's one way to distribute them:



If you have four yellow (Y) cubes, four green (G) cubes, two blue (B) cubes, and two magenta (M) cubes, you could distribute them this way:



The result you're proving in this problem forms the basis for the alias method, a fast algorithm for simulating rolls of a loaded die. This has applications in machine learning (simulating different outcomes of a random event), operating systems (allocating CPU time to processes with different needs), and computational linguistics (generating random sentences based on differently-weighted rules).

* This problem adapted from Exercise 3.4.1.7 of *The Art of Computer Programming, Third Edition, Volume II: Seminumerical Algorithms* by Donald Knuth.